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On gonihedric loops and quantum gravity

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Abstract
We present an analysis of the gonihedric loop model, a reformulation of the two-dimensional gonihedric spin model, using two different techniques. First, the usual regular lattice statistical physics problem is mapped onto a height model and studied analytically. Second, the gravitational version of this loop model is studied via matrix models techniques. Both methods lead to the conclusion that the model has $c_{\text{matter}} = 0$ for all values of the parameters of the model. In this way it is possible to understand the absence of a continuous transition.

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1. Introduction

The gonihedric string model (a type of string model with vanishing tension at tree level) was introduced and elaborated further in [1]. The proponents argued its potential relevance to describe the QCD string and is, in any case, an interesting model of random surfaces. The so-called gonihedric spin model provides a dual version of the gonihedric string embedded in a (hyper)cubic discrete space [2, 3]. This spin model describes the gonihedric string as the interface between positive and negative spin regions.

The thermodynamical properties of this spin model have been already considered before. The existence of a second-order phase transition separating the 'ordered' and 'disordered'[1] phases at $T = T_c$ would make it possible to describe the continuum properties of the discrete gonihedric string. Several papers studied the phase diagram of this model [5, 6], but the basic version of the gonihedric spin model exhibited a first-order phase transition rather than a continuous one. This led the inclusion of a new parameter κ [3] that regulates the self-avoidance of the 'string'. For $\kappa = 0$, the original non-self-avoiding string theory is recovered, while for increasing κ the string becomes more and more self-avoiding. Of course a non-zero value for κ is probably of physical interest anyway. With the addition of this new parameter,

[1] The quotation marks are appropriate here as this model has an exponentially large degeneration and so what order and disorder means is not completely apparent, see [4].

the phase diagram changed considerably exhibiting second-order phase transition for κ greater than some κ_c.

At some point, the spin model gained interest by itself. Extremely slow approximation to the equilibrium in a region below T_c strongly suggested the presence of glassiness in the dynamics of this three-dimensional model. The model was studied in [7]. It revealed indeed the presence of glassiness, when $\kappa = 0$, in its dynamics for $T < T_g$, where T_g ($T_g < T_c$) is the glassy temperature. T_g was determined with good precision in the model. For $\kappa \neq 0$, it was seen that the first-order transition present at $\kappa = 0$ persists up to some κ_c. From this point on the transition is of second order. Also T_g tends to T_c for $\kappa \neq 0$ up to a point where it becomes impossible to tell them apart; for big enough κ the glassy phase extends all the way up to T_c. The dynamics of a family of very similar models (but not including the gonihedric model for any value of κ) had already been studied in [8].

The very geometrical origin of this spin model fine tunes the coupling constants (only nearest neighbours, next to nearest neighbours, and plaquette-like operators appear) to very concrete values. With the addition of the self-avoidance κ parameter, a one-parameter family of spin models survives. In particular, the energy of a spin configuration can be seen to be

$$E = n_2 + \kappa n_4, \tag{1}$$

where n_2 is the length of corners of the inter-phase separating plus and minus spins, and n_4 is the length of the self-crossings of this inter-phase, all in the lattice units. This is in fact the energy of the gonihedric string, where the flat regions of the string does not contribute to it at all.

These fine-tuned coupling constants are the reason for some crucial features. The Hamiltonian exhibits a large amount of symmetry; one can for instance flip at once any whole plane of spins (in three dimensions) in any direction without changing the energy of the configuration (some of this symmetry is lost when $\kappa \neq 0$). This feature is thought to be relevant in the appearance of the glassiness [7]. The appearance of energy barriers along the evolution of the spin system is another determinant characteristic of this system which is certainly responsible for very slow dynamics; some metastable states are formed where the system gets trapped for a long time.

In [9], the model was reduced to two dimensions and the dynamics in the two-dimensional model studied. Equation (1) remains valid and now n_2 corresponds to the number of corners and n_4 the number of crossings of the boundary line separating plus and minus domains. This two-dimensional model can actually be represented by a model of loops. We shall term this as the gonihedric loop model.

Although the two-dimensional model exhibits large degeneracy and trapped, long lived, metastable states, the dependence of the barriers on the linear scale of the system is different and no glassiness was found in this case. Only a very slow, power-like, but certainly faster than logarithmic, evolution to equilibrium was found. An analytical estimation of the power-like behaviour was computed and found to be in a reasonable accordance with numerical simulations.

As for the thermodynamical properties, the two-dimensional model seems to have no criticality at all. It is in fact solvable and trivial for $\kappa = 0$ (it actually corresponds to one of Baxter's six vertex models [10]), while there is no analytical solution for other κ values.

In this work, we plan to study with different analytical techniques the gonihedric loop model, both on its own and coupled to gravity. In this latter version the underlying lattice is dynamical.

In section 2, we briefly recall the basic definitions of the gonihedric loop model. In section 3, we study the gonihedric loop model in two dimensions in the framework set by

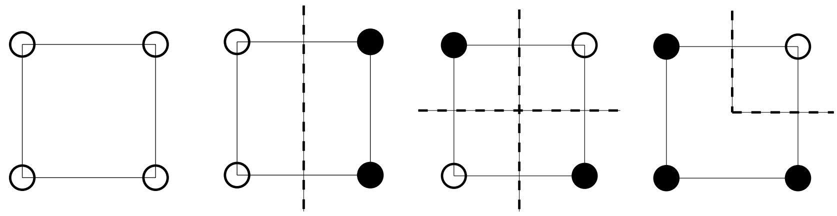

Figure 1. All kinds of plaquette terms up to symmetries and its corresponding loop segment. (a) and (b) contribute with some e_0 energy while (c) contributes with 4κ and (d) with 1 to the energy.

Kondev *et al* in [11] and by Di Francesco and Guitter in a series of papers (see [12] for a review and references therein). These techniques are developed to be used in loop models with no temperature or energy parameter; they correspond to a combinatorial problem. With some heuristic arguments we will connect their techniques and results with our model.

In section 4, we couple the gonihedric loop model to gravity. This can be performed using random matrix mode techniques, more precisely we will write it in terms of a Hermitian two-matrix model. This new model is already interesting by itself as it has not been solved exactly before (although some very similar models have been, see [13]). The peculiar interactions between the two matrices with independent coupling constants makes it very hard to solve. Anyway this will provide us an analytical handle on the gonihedric loop model on randomly generated lattices. In the limit $\kappa \to \infty$, one of the matrices in the matrix model reduces to a Gaussian variable that can be exactly integrated out. Unfortunately this does not make the model directly solvable. To test then the heuristic arguments we have alluded to before we study the renormalization group flow of the coupling constants using the methods in [14]. As a byproduct we find again confirmation of the non-existence of a continuous phase transition in the gonihedric spin model through the KPZ relations [15].

2. The gonihedric loop model

First of all we shall present how to construct the loop configurations from the gonihedric spin configurations and how to identify the points where energy is located.

The gonihedric spin model Hamiltonian has the following form in two dimensions:

$$\mathcal{H}^{2\mathrm{D}}_{\mathrm{gonih}} = -\kappa \sum_{\langle i,j\rangle} \sigma_i\sigma_j + \frac{\kappa}{2} \sum_{\langle\langle i,j\rangle\rangle} \sigma_i\sigma_j - \frac{1-\kappa}{2} \sum_{[i,j,k,l]} \sigma_i\sigma_j\sigma_k\sigma_l. \tag{2}$$

When computing the energy we realize that there are only three possible levels of energy for a plaquette. In figure 1, we can see all possible configurations up to rotation and Z_2 symmetries. As we can see in the figure we can associate with each spin configuration the inter-phase configuration just drawing a line (dashed line) that separates plus and minus spins. This inter-phase will be the gonihedric loop and the energy will be accumulated in configurations where the loop bends or crosses with itself.

Using the Hamiltonian in equation (2), we compute the energy costs to generate bendings and crossings. While figure 1(a) and (b) cost some bulk energy e_0, figure 1(c) costs $e_0 + 4\kappa$ and (d) costs $e_0 + 1$ in energy units. That means that all the energy will accumulate in regions where the loop is bended or it crosses with itself. Then, apart for a configuration-independent term proportional to the volume, we get an energy per configuration given by equation (1).

The Hamiltonian (2) thus describes a theory of loops defined on a two-dimensional lattice, modulo the usual Z_2 degeneracy of Ising-like models. The weight of each loop configuration is given by formula (1). This is the Gonihedric loop model.

Particularly for $\kappa \neq 0$, the model looks like an ordinary Ising model with some additional interactions. This similarity is in fact way too naive. In fact, the critical behaviour as studied in the references already mentioned is quite different. In this work we shall understand why.

In fact, when placed on a square regular lattice, the above statistical model is trivial for $\kappa = 0$ (only plaquette interactions survive in this case, and we obtain a model quite different from Ising). For $\kappa \neq 0$, there is no known exact solution of the model. When we place the above Hamiltonian on a randomly generated lattice, it will correspond to a model of two-dimensional gravity possibly with some matter contents. The former case is studied in the next section while the latter is studied in section 4.

3. Relation to height models

In a series of works (see [12] for a review) several aspects of loop models in connection with folding and meander problems have been studied. There is a mapping from loop models onto the so-called height models. These height models are going to be interesting to us because they provide a systematic way to conjecture the actual value of the central charge c. These results are actually based on a previous work by Kondev *et al* [11] where the continuous effective Coulomb gas formulation is studied.

Let us describe the height models for different classes of loops and recall some of the results and formalism by Kondev *et al*. We assign to each loop a weight $n = 2\cos(\pi e)$. The $e = 0$ case ($n = 2$) corresponds to the oriented loop model[2]. The case $e = \frac{1}{3}$, corresponding to a weight $n = 1$, corresponds to non-oriented loops. Let us work first with this $e = 0$ case, i.e. oriented loops.

3.1. Fully- and dense-packed loop models on a trivalent regular lattice

Let us begin by considering the so-called fully-packed loop model on a regular trivalent lattice. This section is based on [12]. Fully-packed means that each site is visited once and only once by a loop. We will call c_{FPL} the central charge of this model. An example of such a loop configuration is drawn in figure 2. The bicolourability of nodes of the lattice enables us to attach labels on the loop unit segments. This is equivalent to saying that every loop has an even number of unit segments—then they can be labelled using two labels alternatively. We call B the segment joining a black node to a white node following the loop orientation, and C the segment joining a white node to a black one following the loop orientation. We will give a label also to the unvisited edges, let us say A. In figure 2, this label association is represented. Once we have defined the different types of unit segments in our theory we will define the associated height theory.

Define a height X variable in the centre of every face. This height variable X will differ from one face to another, it actually increases by a defined quantity each time we cross a link. This quantity is defined following the rules shown in figure 3. Now we just need to ensure that the height variable is globally defined, which means that the total gained height in any closed path is zero. This amounts in this case to the condition $A + B + C = 0$, i.e. two real degrees of freedom remain and so we can describe the model with a two-dimensional local

[2] A weight $n = 2$ may be understood as two possible equally probable states for each loop, this could be encoded in their orientation.

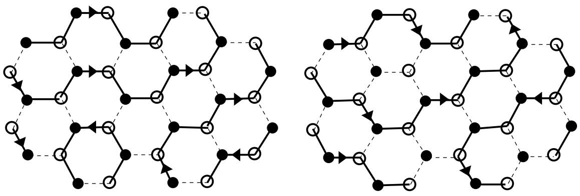

Figure 2. An example of an oriented-loop model configuration on a bicolourable regular trivalent lattice. The left one is for the fully-packed model while the right one is for the dense-packed model. The corresponding central charges are c_{FPL} and c_{DPL}.

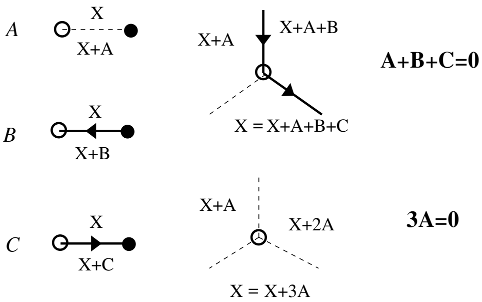

Figure 3. Rules defining the increments of the height variable when crossing an edge (left) and condition for the height increment unit vectors A, B and C when going around a vertex (right). The bottom one is present only in the dense-packed loop model.

field. This implies in the continuum limit a model with two scalar fields (the two components of the height field) and so the central charge for the fully-packed loop model $c_{\mathrm{FPL}} = 2$.

A dense-packed loop model is one where the condition of each site being visited by a loop is relaxed. In the case of dense-packed loop model, we need to consider a new kind of vertex. This new vertex is just the non-visited vertex, the vacancy or impurity vertex. The condition we find after walking around a non-visited vertex is $A = 0$, thus $B = -C$, and so the associated height model reduces to a one-dimensional height variable. In the continuum limit, the associated continuum height model contains only one scalar height field and so in the dense case the central charge for the dense-packed loop model is $c_{\mathrm{DPL}} = c_{\mathrm{FPL}} - 1 = 1$. Note that the $B = -C$ condition erases all information about the bicolourability of the lattice. We can see in figure 3 that when $B = -C$ it does not matter what colour the nodes are.

This was treated in general in the works of Kondev, de Gier and Nienhius [11], and used by Di Francesco and Guitter [12]. In the general picture a weight $n = 2\cos(\pi e)$ per loop ($e \in [0, 1]$) is introduced and the model is studied following an effective field theory description. This analysis gives

$$c_{\mathrm{DPL}} = c_{\mathrm{FPL}} - 1 = 1 - 6\frac{e^2}{1-e}. \tag{3}$$

For the non-oriented case, equivalent as we have already said to $e = \frac{1}{3}$ in the general description, we can still analyse the height components in the same form. We have no orientation so we cannot use the bicolourability information to define in a meaningful way the labels on the segments. Then we get only two kinds of height increments, say A and B. In figure 4, we can see that there is no way to distinguish different kinds of segments on the loop. The bicolourability property combines with the orientable nature of the loops to define the B

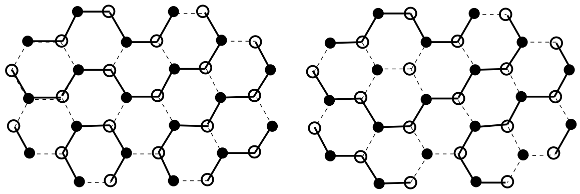

Figure 4. An example of a non-oriented loop model configuration on a bicolourable regular trivalent lattice. Fully-packed loop model on the left and dense-packed loop model on the right.

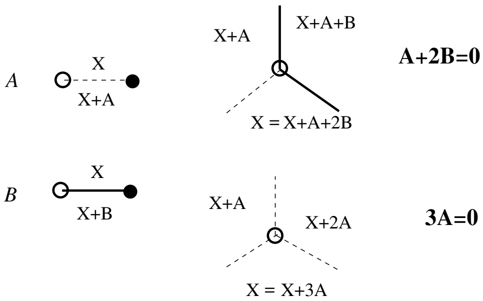

Figure 5. Left: in this case the height increment rules involve only two increment vectors, A and B. Right: the conditions on this vectors.

and C increment vectors. The loss of any of the two ingredient forces a constraint between B and C. The rules for the height increments are shown in figure 5.

In the fully-packed non-oriented loop model, the only vertex we have is again a visited vertex with two loop segments and an empty edge. The condition ensuring a meaningful height variable is thus $A + 2B = 0$. Following again the same steps as before we end up with a one-dimensional height variable, meaning that $c_{\mathrm{FPL}} = 1$ in the non-oriented loop model on a trivalent regular lattice.

The dense-packed non-oriented loop model is again straightforward. The extra condition for the new unvisited vertex is $3A = 0$. This constraints A and B to zero so the height variable remains constant in the whole lattice. This means no scalar field in the continuum theory, and consequently $c_{\mathrm{DPL}} = c_{\mathrm{FPL}} - 1 = 0$. Note that this coincides with the values we would obtain from equation (3) when applying $e = \frac{1}{3}$.

These two examples help us to understand how the height models are constructed, and how to extract the information we are looking for.

3.2. Non-oriented dense loops on tetravalent regular lattice

Taking the gonihedric spin model as our starting point, we can arrive at a loop model similar to the ones we have just described. One difference is that our spin model is placed in a square lattice and so are the loops that live on the dual lattice. As already explained above these loops will add energy to the system due to bending ($\Delta E = 1$) and crossing ($\Delta E = 4\kappa$).

Let us consider first the case where $\kappa = \infty$ (infinite self-avoidance). In this case there is no crossing of the loops, so if we forget for a moment about the temperature we have a model of dense loops with no self-crossings. They are of course not oriented loops since there is no way the spin configuration can induce an independent orientation to each loop. The case where $\kappa < \infty$ can be recovered from the $\kappa = \infty$ case just by relaxing the no crossings

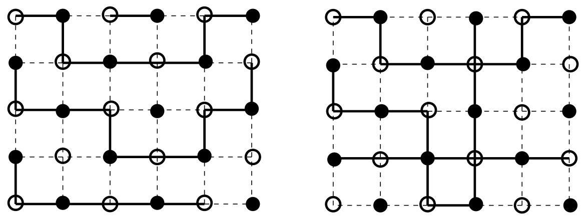

Figure 6. An example of a dense-packed non-oriented loop model configuration on a bicolourable regular tetravalent lattice. On left-hand side there is a non-self-intersecting loop model while right-hand side the example shows a configuration with self-crossing loops.

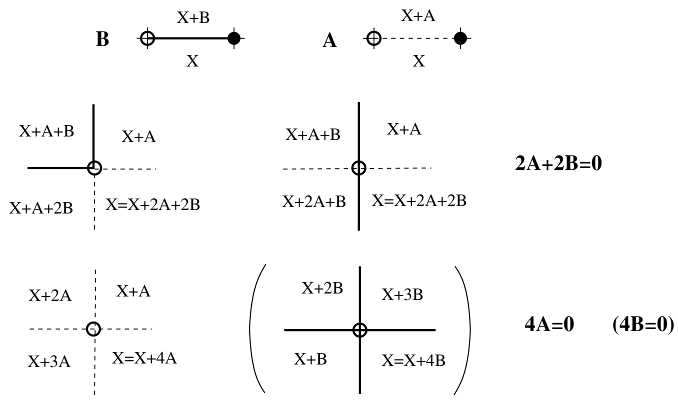

Figure 7. Rules defining the height increments from the loop configuration and consistency equations from the different kind of vertex. In parenthesis the vertex is appearing with the self-intersections.

condition. In the opposite limit ($\kappa = 0$) crossing configurations occur freely. We will argue later how to extract conclusions for the whole range of κ.

In figure 6, we see loop configurations with and without intersections. The rules needed to label all (occupied or not) edges, and the possible vertex present in our model are shown in figure 7. Since our loops are not oriented we cannot define in a consistent way the alternated labelling of the segments of the loop. This is why all segments in the loop have to have the same label B. In the same way, considering the 'dual loop' constructed from the unvisited edges, we see easily that all unvisited vertexes have the same label $A \neq B$. Now, looking at the total height difference when going around one site in our lattice, we can extract some consistency conditions for our height unit increments A and B. These local consistency conditions are depicted in figure 7. It is easy to see then that the equations required from consistency of the height variables

$$2A + 2B = 0 \tag{4}$$

$$4A = 0 \tag{5}$$

$$4B = 0 \tag{6}$$

constrain our height variable so much that it is in fact constant all over the lattice.

We should remember that there is no temperature or Hamiltonian in these loops models from which we extract the height model. The connection with our model appears when we

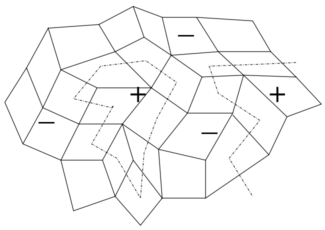

Figure 8. Example of a random lattice with a gonihedric spin model on it.

consider the relevance of the bending vertex. We obtain the same equations for A and B either allowing or not the bending of the loops. This corresponds to studying our loops inherited from the gonihedric spin model at $T = 0$ or $T = \infty$, respectively[3]. We should then come to the conclusion that there is no continuous transition at finite temperature since both extrema share the same critical behaviour [16]. The same reasoning with the self-crossing vertex being allowed or not brings us to the same conclusion for every value of the coupling constant κ in the gonihedric Hamiltonian[4]. Thus we end up again in a $c = 0$ theory in the whole κ space. This means no criticality, in perfect agreement with numerical simulations [9].

4. The two-dimensional gonihedric spin model coupled to gravity

We are now going to couple the gonihedric loop model to gravity. This is interesting by itself and in order to understand the effect of placing the gonihedric spin model on a 'fluid' surface. Of course the resulting model is nothing but the one-dimensional version of the gonihedric string in a two-dimensional embedding curved dynamical space[5].

The energy of the gonihedric spin model can still be written, on a fixed quadrangulation, as $n_2 + 4\kappa n_4$, where n_2 is the number of unit 'corners' of the inter-phase between plus and minus sign spins, and n_4 is the number of self-crossing points of the inter-phase.

In order to couple this model to gravity we need to put our spins in the sites of a random lattice built from square pieces[6]. We will consider, when coupling the model to gravity, the limiting case where the loops never cross themselves. This non-intersecting limit corresponds to the $\kappa \to \infty$ limit in the gonihedric spin action. In figure 8, we can see an example of this kind of quadrangulations.

To describe these configurations we shall introduce $N \times N$ Hermitian random matrices. Let us see how to do this. We have first of all plaquettes where no loop goes across them, then plaquettes crossed by one loop without bending through it (a straight loop piece across a unit square of the lattice), and finally plaquettes were the loop crossing them bends 'right' or 'left'. These three building blocks of the random lattices are presented graphically[7] in figure 9 with the corresponding term of the associated matrix model that will generate them upon integration. As we can see we are considering the most general case where all the couplings

[3] Note that in the loop/height model, a bending vertex has the same weight as the straight loop vertex and the bulk. This corresponds to infinite temperature. When we do not allow bending to appear, it is equivalent to freezing the gonihedric loops at $T = 0$.

[4] Remember that $\kappa = 0$ is in fact trivially solved [10] and does not possess any thermodynamical transition.

[5] In higher-dimensional cases related studies in the continuum have been performed by [17].

[6] We chose squares as the building blocks of our random lattice since our original model is defined in a square regular lattice. It happens to be important in order to define what exactly means bending the loop.

[7] To simplify the visual identification with the loop model, we have not drawn the lines corresponding to the bulk propagator, i.e. to the A matrix propagator. The loop is generated with the B matrix propagator.

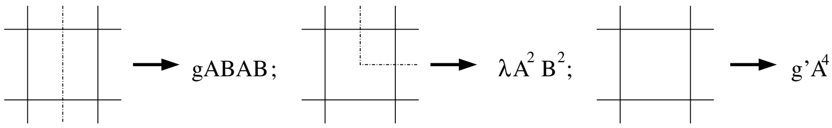

Figure 9. Correspondence between the loop pieces and the matrix interaction that are going to generate them.

are different. In our specific case we will impose the condition $g' = g$ due to the fact that a straight piece of loop does not contribute with any amount of energy to the action, so the coupling needs to be equal to the bulk coupling. In addition to these 'interaction' terms we need to include the kinetic term for the two matrices, i.e. the quadratic terms A^2 and B^2 that generates the propagators. So finally the matrix model that will represent our gonihedric loop model coupled to gravity is

$$Z_N(g,\lambda) = \int^{(N)} \mathrm{d}A\,\mathrm{d}B\exp\left[-N\,\mathrm{Tr}\left(\frac{1}{2}(A^2+B^2)+\frac{g}{4}A^4+\frac{g}{2}ABAB+\lambda A^2B^2\right)\right].$$

The numerical factors preceding each interaction term are in fact symmetry factors. To our knowledge this model has not been solved exactly. Although somewhat similar matrix models have been solved elsewhere [13], their solution cannot be applied directly to our matrix model.

The coupling constants g, g' and λ have a definite meaning in terms of the statistical model we are treating. $g = g'$ are in fact representing the bulk energy of the unit square, i.e. it represents the conjugate variable to the area of the lattice, the cosmological constant. On the other side, the combination $\frac{\lambda}{g}$ is equal to the statistical Boltzmann weight of each bended loop piece that contains all the information from the energy of the statistical model:

$$g = g' = \frac{\mathrm{e}^{-\beta E_{\mathrm{bulk}}}}{N}, \qquad \frac{\lambda}{g} = \mathrm{e}^{-\beta E_{\mathrm{bend}}}. \tag{7}$$

4.1. Partial integration

The B matrix integration is in fact a Gaussian so it can be integrated exactly. This integration was first presented in [18]. We first diagonalize the A matrix

$$\begin{aligned} Z_N(g,\lambda) &= \int \mathrm{d}A\,\mathrm{d}B\exp\left[-N\,\mathrm{Tr}\left(\frac{1}{2}(A^2+B^2)+\frac{g}{4}A^4+\frac{g}{2}ABAB+\lambda A^2B^2\right)\right] \\ &= \int\left(\prod_{i=1}^{N}\mathrm{d}a_i\,\mathrm{d}B_{ii}\right)\left(\prod_{j>i=1}^{N}\mathrm{d}\,\mathrm{Re}\,B_{ij}\,\mathrm{d}\,\mathrm{Im}\,B_{ij}\right)\prod_{j>i=1}^{N}(a_j-a_i)^2\,\mathrm{e}^{-NS(\{a_i\},\{B_{ij}\})} \end{aligned} \tag{8}$$

where $\{a_i\}$ are the eigenvalues of A. The action $S(\{a_i\},\{B_{ij}\})$ is

$$S(\{a_i\},\{B_{ij}\}) = \sum_{i=1}^{N}\left(\frac{1}{2}a_i^2+\frac{g}{4}a_i^4\right)+\frac{1}{2}\sum_{i=1}^{N}B_{ii}^2\big(1+(g+2\lambda)a_i^2\big) \tag{9}$$

$$+\sum_{j>i=1}^{N}\big(\mathrm{Re}\,B_{ij}^2+\mathrm{Im}\,B_{ij}^2\big)\big(ga_ia_j+\lambda\big(a_i^2+a_j^2\big)\big). \tag{10}$$

Rescaling the B matrix entries correspondingly we transform the B integral in a bunch of decoupled Gaussian integrals

$$Z_N(g,\lambda) \sim \int \prod_{i=1}^{N} \mathrm{d}a_i \prod_{j>i=1}^{N} (a_j - a_i)^2 \prod_{i,j=1}^{N} \left(\delta_{ij} + g a_i a_j + \lambda\left(a_i^2 + a_j^2\right)\right)^{-1/2} \mathrm{e}^{-N\sum_{i=1}^{N}\left(\frac{1}{2}a_i^2 + \frac{g}{4}a_i^4\right)}$$
$$= \int \mathrm{d}A \left(\mathrm{Det}[\mathbb{I}\otimes\mathbb{I} + gA\otimes A + \lambda(A^2\otimes\mathbb{I} + \mathbb{I}\otimes A^2)]\right)^{-\frac{1}{2}} \mathrm{e}^{-N\,\mathrm{Tr}\left(\frac{1}{2}A^2 + \frac{g}{4}A^4\right)} \qquad (11)$$

where Det means that a determinant of a tensor product matrix, i.e. a $N^2 \times N^2$ matrix.

This is a nice expression of the two matrix model as a one matrix model. Although some especial cases of this model with specially fine-tuned coupling constants have been solved, it appears to be very difficult to solve in the general parameter setting which is interesting to us. So we have to turn to other techniques.

5. The renormalization group approach

The renormalization group approach for random matrices was first introduced in [14] by Brézin and Zinn-Justin and later improved by Higuchi, Itoi, Nishigaki and Sakai in [19]. The improvement in [19] provides a more quantitative analysis of the critical structure in parameter space. It consists basically in using the equations of motion of the matrix model (also called loop equations) in order to deal with redundant operators in the renormalized action in the large N limit. This more complete analysis is much more involved and we have not applied it to our model but instead used the simpler Brézin–Zinn-Justin approach since it is equally valid to get the qualitative flow we are looking for.

The renormalization procedure in matrix models consists in the integration of the last row and column of the matrices, so that we relate the partition function $Z^{(N+1)}$ to $Z^{(N)}$. One could in principle do so with the one-matrix reduced model (11), but due to non linearities in $\mathrm{Tr}(A)$ and $\mathrm{Tr}(A^2)$ already in the saddle-point equation we decided to return to the original two-matrix model, where this non-linearity is not present. In order to make easier the 'last-row–last-column' integration we are going to write the matrices A and B as follows:

$$A_{N+1} = \begin{pmatrix} A_N & 0 \\ 0 & \alpha \end{pmatrix} \qquad (12)$$

$$B_{N+1} = \begin{pmatrix} B_N & \vec{v} \\ \vec{v}^{\dagger} & \beta \end{pmatrix}. \qquad (13)$$

Since the action is Gaussian in the B matrix, we know that the integral in the $\vec{v}$ vector and that of the β variable will be easy to perform. Only a saddle point in the α variable will be left as in the original approach [14].

To perform the renormalization group computations we need to free our parameters from the constraint that the original statistical model imposes on them, i.e. we should let $g \neq g'$. The partition function will satisfy

$$Z_{N+1}(g, g', \lambda) = C(N)^{N^2} Z_N(g+\delta g, g'+\delta g', \lambda+\delta\lambda) \qquad (14)$$

where δg, $\delta g'$ and $\delta\lambda$ depend on N. This equation will be well approximated by the linear differential equation

$$\left[N\frac{\partial}{\partial N} - \beta_g\frac{\partial}{\partial g} - \beta_{g'}\frac{\partial}{\partial g'} - \beta_\lambda\frac{\partial}{\partial\lambda} + \gamma\right] F(N, g, g', \lambda) = r(N), \qquad (15)$$

where $r(N)$ is related to $C(N)$ and

$$F(N, g, g', \lambda) = -\frac{1}{N^2} \log \left[\frac{Z_N(g, g', \lambda)}{Z_N(0, 0, 0)} \right] \tag{16}$$

is the free energy of our model.

5.1. Last-column–last-row integration

Consider

$$Z_{N+1}(g, g', \lambda) = \int^{(N+1)} \mathrm{d}A\, \mathrm{d}B \times \exp\left(-(N+1)\, \mathrm{Tr}_{N+1} \left[\frac{1}{2}(A^2 + B^2) + \frac{g'}{4} A^4 + \frac{g}{2} ABAB + \lambda A^2 B^2 \right] \right).$$

We rotate both matrices with the same $U \in U(n)$ unitary matrix in such a way that the A matrix diagonalize. One unitary integral decouples and contributes with a factor $V_{U(N+1)} = \pi^{N(N+1)/2} \big/ \prod_{p=1}^{N+1} p!$. The matrix A is now

$$A_{N+1} = \begin{pmatrix} a_1 & & & 0 \\ & \ddots & & \\ & & a_N & \\ 0 & & & \alpha \end{pmatrix}.$$

After this we rotate back the first N rows and columns so that A and B will look like equation (13).

It is easy to see that all the interaction terms in the action can be written as

$$\begin{aligned}
\mathrm{Tr}_{N+1}[A^n] &= \mathrm{Tr}_N[A^n] + \alpha^n \\
\mathrm{Tr}_{N+1}[B^2] &= \mathrm{Tr}_N[B^2] + \beta^2 + 2\vec{v}^\dagger \vec{v} \\
\mathrm{Tr}_{N+1}[ABAB] &= \mathrm{Tr}_N[ABAB] + 2\alpha \vec{v}^\dagger A \vec{v} + \alpha^2 \beta^2 \\
\mathrm{Tr}_{N+1}[A^2 B^2] &= \mathrm{Tr}_N[A^2 B^2] + \alpha^2 (\vec{v}^\dagger \vec{v} + \beta^2) + \vec{v}^\dagger A^2 \vec{v}.
\end{aligned}$$

After substituting this into the partition function we find several terms. Those depending on $\vec{v}$, β and α will be integrated out so that a new effective action for the $N \times N$ two-hermitian matrix model will arise. After performing the integrations on $\vec{v}$ and β, we find

$$Z_{N+1}(g, g', \lambda) = \frac{V_{U(N+1)}}{V_{U(N)}} C_N \int^N \mathrm{d}A\, \mathrm{d}B \exp\left(-(N+1)\, \mathrm{Tr}_N \times \left[\frac{1}{2}(A^2 + B^2) + \frac{g'}{4} A^4 + \frac{g}{2} ABAB + \lambda A^2 B^2 \right] \right) I(g, g', \lambda), \tag{17}$$

where

$$C_N = \left(\frac{\pi}{N+1} \right)^N \left(\frac{2\pi}{N} \right)^{\frac{1}{2}}, \tag{18}$$

and

$$I(g, g', \lambda) = \int \mathrm{d}\alpha \frac{\det[A - \alpha]^2}{\det[1 + \lambda\alpha^2 + g\alpha A + \lambda A^2](1 + (2\lambda + g)\alpha^2)^{\frac{1}{2}}} \times \exp\left(-(N+1) \left(\frac{\alpha^2}{2} + \frac{g'}{4} \alpha^4 \right) \right). \tag{19}$$

We can already see here that we are not going to receive any direct contribution to the renormalization of the λ or g constant in the action. There is no B matrix dependence in the saddle-point integral, so we cannot generate the operators $\mathrm{Tr}(A^2B^2)$ or $\mathrm{Tr}(ABAB)$ in our renormalization scheme. The only flow of these two parameters comes from the renormalization of the 'kinetic' $\mathrm{Tr}(A^2)$ operator after field rescaling. The α integral cannot be computed explicitly. The saddle-point method has been used to solve it in the large N limit. The two determinants in the integrand contribute to the saddle point while the square root in the denominator does not.

5.2. *Saddle-point equation*

The saddle-point equation reads

$$\frac{2}{N}\,\mathrm{Tr}\left(\frac{1}{\alpha_s - A}\right) = \left(\alpha_s + g'\alpha_s^3\right) + \frac{1}{N}\,\mathrm{Tr}\left(\frac{2\lambda\alpha_s + gA}{1+\lambda\alpha_s^2 + g\alpha_s A + \lambda A^2}\right). \tag{20}$$

To solve this saddle-point equation, we will assume that α_s, the solution of the saddle-point equation, has an expansion in powers of $\hat{a}_j = \frac{1}{N}\mathrm{Tr}(A^j)$. Once this saddle point is computed we must introduce α_s into the integral, then we will again expand everything to find the contribution of this integral to the operator $\hat{a}_2$

$$I(g, g', \lambda) = \frac{\det[A-\alpha_s]^2}{\det\left[1+\lambda\alpha_s^2 + g\alpha_s A + \lambda A^2\right]\left(1+(2\lambda+g)\alpha_s^2\right)^{\frac{1}{2}}}\exp\left(-(N+1)\left(\frac{\alpha_s^2}{2} + \frac{g'}{4}\alpha_s^4\right)\right) \tag{21}$$

$$= \exp -(N+1)\left(C_0 + C_1\hat{a}_1 + C_{1,1}\hat{a}_1^2 + C_2\hat{a}_2 + \cdots\right) \tag{22}$$

To keep track of the terms we need in the expressions we will perform the change of variables $A \to \epsilon A$. At the end of the calculation we shall put $\epsilon = 1$:

$$\frac{2}{N}\,\mathrm{Tr}\left(\frac{1}{\alpha_s - \epsilon A}\right) = \left(\alpha_s + g'\alpha_s^3\right) + \frac{1}{N}\,\mathrm{Tr}\left(\frac{2\lambda\alpha_s + g\epsilon A}{1+\lambda\alpha_s^2 + g\alpha_s\epsilon A + \lambda\epsilon^2 A^2}\right). \tag{23}$$

We are going to expand everything up to second order in ϵ. This will be enough to extract the C_2 coefficient. We expand

$$\alpha_s = \alpha_s^{(0)} + \epsilon\tilde{\alpha}_s^{(1)} + \epsilon^2\tilde{\alpha}_s^{(2)} + \mathcal{O}(\epsilon^3), \tag{24}$$

where

$$\tilde{\alpha}_s^{(1)} = \alpha_s^{(1)}\hat{a}_1, \qquad \tilde{\alpha}_s^{(2)} = \alpha_s^{(2)}\hat{a}_2 + \alpha_s^{(1,1)}\hat{a}_1^2. \tag{25}$$

At order $\mathcal{O}(\epsilon^0)$, we get an equation for $\alpha_s^{(0)}$

$$\alpha_s^{(0)} + g'\alpha_s^{(0)3} - \frac{2}{\alpha_s^{(0)} + \lambda\alpha_s^{(0)3}} = 0 \tag{26}$$

that is a sixth-order equation reducible to a third order one. Of all solutions we will choose the one that matches the known solution for $g \to 0$ and $\lambda \to 0$, that is $\alpha_s^{(0)} \to \sqrt{2}$ [8] and which is real. At next order we get linear equations for $\alpha_s^{(1)}$, and at order $\mathcal{O}(\epsilon^2)$ we get two linear equations. We have tested the stability of the saddle-point solution we have found in the vicinity of the point $g = \lambda = 0$ using the above expansion. We have checked that the solution is stable for small values of these coupling constants in the whole g' range.

At this point we can express the saddle-point evaluation of the integral in terms of these constants and find the renormalization of the operator $\mathrm{Tr}(A^2) = N\hat{a}_2$. If we call Z_2 the

[8] We could also choose $-\sqrt{2}$.

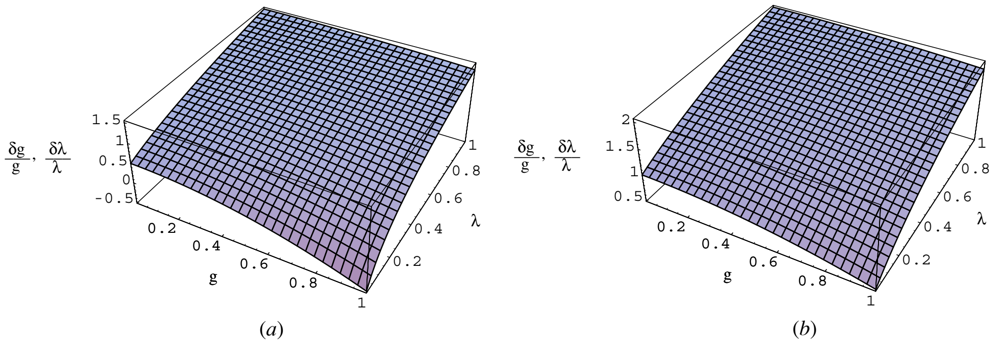

Figure 10. $\frac{\delta g}{g}$ and $\frac{\delta\lambda}{\lambda}$ for positives values of the coupling constants around the origin (physical quadrant). Two different values of g' have been plotted to show the positiveness of this quantity for all g'. (*a*) $g' \sim 0$. (*b*) $g' = 1$.

coupling relative to that operator (when we begin $Z_2 = 1$) the renormalized coupling Z_2^{ren} after one renormalization group step from the original model will be

$$Z_2^{\text{ren}} = \frac{N+1}{N}\left(1 + \frac{1}{N+1}C_2\right), \tag{27}$$

where

$$C_2 = \frac{2 + 6\alpha_0^2 - (g^2 - 4\lambda^2)\alpha_0^4}{2\left(\alpha_0 + \lambda\alpha_0^3\right)^2} \tag{28}$$

and α_0 is the solution we took from the sixth-order equation (26).

5.3. *Renormalization-group flow*

We must scale the A matrix in order to keep $Z_2^{\text{ren}} = 1$ at the end of the complete renormalization group process. This induces a flow on the couplings λ and g.

$$\lambda^{\text{ren}} = \lambda\left(1 - \frac{1}{N}C_2 + \mathcal{O}(N^{-2})\right) \tag{29}$$

$$g^{\text{ren}} = g\left(1 - \frac{1}{N}C_2 + \mathcal{O}(N^{-2})\right) \tag{30}$$

The behaviour under the renormalization-group depends on the sign of C_2.

If we pay attention to the statistical meaning of the couplings in the original model, it is clear that we can keep $g \ll 1$ and both g and λ/g are positive, i.e. $\lambda, g > 0$ is the physical quadrant. As in the usual renormalization group approach $\frac{\delta(g-g^*)}{g-g^*} < 0$ near $g = g^*$ implies an infrared fixed point at $g = g^*$. This is the case in the physical quadrant as can be seen in figure 10.

In other words, the two constants λ and g renormalize towards its fixed value $\lambda = g = 0$. Thus only the g' coupling constant remains. The constant g' accompanies is the operator $\text{Tr}(A^4)$ which generates only pure gravity critical behaviour. This means $c_{\text{matter}} = 0$.

We can also think of this flow in the following way. Regardless of g, if λ is flowing towards zero, we can forget about the corresponding operator and concentrate on the rest of the action. The corresponding model has been solved in [20]. In our special choice of parameters $g' = g$ this brings us to pure gravity again.

The irrelevance of the operator $\mathrm{Tr}(A^2B^2)$ could have also been already predicted with the heuristic tools presented in the first half part of this paper. Using the height model language from [12], we can see that for ordinary gravity[9] the height model we work with contains exactly the same constraints whether we include the $\mathrm{Tr}(A^2B^2)$ operator or not. These arguments show that this operator is not modifying the essential nature of the scalar model living in the continuum. Thus the central charge should be the same.

Although a complete solution of this matrix model has not been achieved, we have been able to explore the renormalization group flow. We find again the conclusion that the gonihedric loop model coupled to gravity corresponds to $c = 0$ central charge for the matter part [15], i.e. pure gravity. This again agrees with the lack of a continuous thermodynamical transition observed for the two-dimensional gonihedric spin model. We, therefore, find a rather nice agreement among all approaches.

6. Conclusions

We have presented two, quite different, versions of the gonihedric loop model.

First, through a direct duality we have transformed our gonihedric spin model into a loop model. We have studied the gonihedric loop model through the height models using the results given in [11, 12]. This height models has been proved to give very satisfactory predictions for the central charge and critical exponents for combinatorial loop models.

We have applied height model technology to the present problem and we have extracted the central charge in some limiting cases of the gonihedric loop model. These limiting cases correspond to zero and infinity temperature, and zero and infinity κ (self-avoidance parameter). Modulo some very plausible hypothesis concerning the interpretation of the central charge and renormalization group flows, we are then able to conjecture the value for the central charge found for the limiting models to the whole temperature and κ space. The value of the central charge happens to be the same for the whole space of parameters ($c = 0$), so we find no continuous transition for the gonihedric model (in agreement with existing numerical simulations [9]).

Secondly, we have coupled our loop model to gravity in the $\kappa \to \infty$ limit. We have worked out a partial integration, since in this limit the *loop* matrix is Gaussian. Unfortunately we could not go further in the exact integration of this matrix model due to its strong non-locality (in eigenvalue space).

A renormalization group approach has proved useful to bypass this problem. We have computed the infinitesimal renormalization flow for the two couplings g and λ. We found that they flow always towards zero, so only a quartic one-matrix model remains. We can also consider only λ as irrelevant coupling constant. The limiting matrix model has been already solved and it turns out to correspond to pure gravity. Then using the KPZ [15] relations, we rediscover the fact that $c = 0$ for the two-dimensional gonihedric spin model.

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[9] Respect to Eulerian gravity, see [12].

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